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COMPARISON OF SQUARED M_{SPLIT} METHOD VS. ABSOLUTE M_{SPLIT} METHOD - CASE STUDY

USPOREDBA KVADRATNE M_{SPLIT} METODE I APSOLUTNE M_{SPLIT} METODE - STUDIJA SLUČAJA

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ABSTRACT

In this paper we analyse and compare two modern methods of deformation analysis: the squared M_{split} method and the absolute M_{split} method. The comparison is made on cases of simulated and actually measured horizontal geodetic networks. The squared M_{split} method is based on the assumptions of the least square method, while the absolute M_{split} method is based on the assumptions of the least absolute deviation method. For each method, we developed a computer programme and used it to calculate point displacements in both a simulated and a measured 2D geodetic network. The purpose of this article is to compare the displacements. The results of the two methods differ by up to about one centimetre in the simulated network and by less than two centimetres in the actually measured geodetic network.

Keywords: squared M_{split} method, absolute M_{split} method, 2D geodetic network, displacement analysis.

SAŽETAK

U ovom radu su analizirane i uspoređene dvije moderne metode deformacione analize: kvadratna M_{split} metoda i apsolutna M_{split} metoda. Usporedba je obavljena kroz slučajeve simuliranih i stvarno izmjerenih horizontalnih geodetskih mreža. Kvadratna M_{split} metoda temelji se na pretpostavkama metode najmanjih kvadrata, dok se apsolutna M_{split} metoda temelji na pretpostavkama metode najmanjeg apsolutnog odstupanja. Za svaku metodu razvijen je kompjuterski program koji je korišten za računanje pomaka tačaka u simuliranoj i izmerenoj 2D geodetskoj mreži. Svrha ovog članka je uporediti dobivene pomake tačaka. Rezultati ove dvije metode razlikuju se za oko jedan centimetar u simuliranoj mreži i manje od dva centimetra u stvarno izmjerenoj geodetskoj mreži.

Ključne riječi: kvadratna M_{split} metoda, apsolutna M_{split} metoda, 2D geodetska mreža, analiza pomaka.

1 INTRODUCTION

The monitoring of engineering structures (bridges, dams, barriers, tunnels, retaining walls, buildings, etc.) is essential for their proper functioning. The analysis of displacements and deformations is based on geodetic measurements of points and sensors embedded in structures or a combination of both (Marković et al., 2019). Displacements and deformations can be

determined between at least two temporal epochs separated by a specific or agreed time interval, which is determined in consultation with the client or the operator of the engineering structure.

Geodetic measurements, including horizontal direction measurements, slope distances, height differences and GNSS vectors, are carried out at points within the geodetic network. These points can be categorised as reference points and points on the structure (Figure 1).

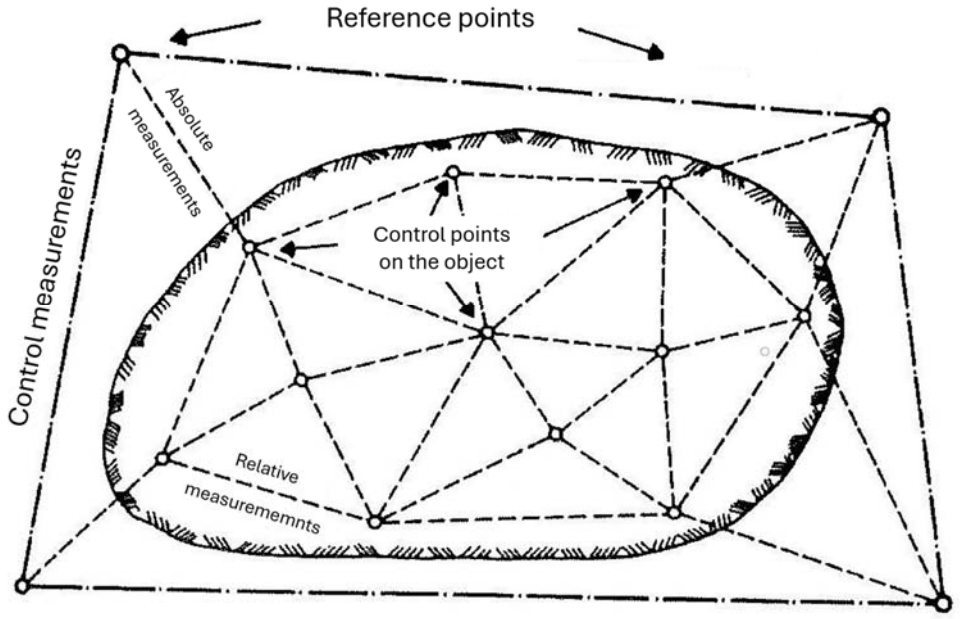

Figure 1. Reference end control points on the object (Pelzer, 1985; Haberler-Weber, 2005).

Points on the structure are points whose position is determined in co-operation with the project planner and which represent potential displacements and deformations of the structure. These points are assumed to be unstable or subject to movement. The reference points, on the other hand, are assumed to remain stationary throughout the operational life of the structure or their displacement over time is precisely known. If the reference points were to move, the coordinate system associated with them would also move. In such a moving coordinate system, it would not be possible to determine the actual displacements of the points on the structure. Therefore, a stable coordinate system realised by stable reference points is essential. Within a stable coordinate system, the actual displacements of the points on the structure can be accurately determined. The identification of stable reference points within a geodetic network is a difficult task. This problem is addressed by deformation analysis.

A classical approach to deformation analysis is the congruence model (Welsch and Heunecke, 2001; Heunecke, 2003). This model compares the geometric state (geometric consistency) of the structure between two measurements (epochs). The input variables for the model are geodetic measurements, while the outputs comprise two groups of points: a group of points that have not moved (stable points) and a group of points that have moved, together with the values of their displacements. The causes of possible displacements and deformations are not considered. Measurements and unknowns are linked with the Gauss-Markov functional model. Several

classical methods are known (Mihailović and Aleksić, 1994): Hannover, Delft, Karlsruhe, München, Fredericton, as well as robust methods and the modern M_{split} methods, which are the focus of this paper.

The M_{split} method was developed and first presented in 2009 by Wiśniewski (Wiśniewski, 2009a, 2009b, 2009c) as a further development of the maximum likelihood method. This approach estimates parameters within two competing functional models without first splitting the observations into subsets. Instead, each observation is automatically assigned to a subset during the iterative process (Wyszkowska and Duchnowski, 2022a). The first variant of the M_{split} method is based on the general assumptions of the least squares method and is therefore referred to as the squared M_{split} method (SMS). This variant has been successfully applied to various geodetic tasks, such as the determination of point displacements in levelling networks (Wiśniewski, 2009a), the detection of gross errors (Wiśniewski, 2009a), coordinate transformation (Janicka and Rapinski, 2023) and the identification of stable reference points in geodetic networks (Nowel, 2019; Zienkiewicz, 2020, 2022). However, a limitation of this variant is its sensitivity to observations that are subject to gross errors and do not match either of the two subsets of observations (Duchnowski and Wiśniewski, 2020). To address this problem, Wyszkowska and Duchnowski introduced a second variant in 2019 that is less sensitive to gross errors (Wyszkowska and Duchnowski, 2019). This variant is based on the general assumptions of the least absolute deviation method (L1 norm condition) and is referred to as the absolute M_{split} method (AMS). It has already been successfully applied in the determination of elevation point displacements (Wyszkowska and Duchnowski, 2019; Duchnowski and Wyszkowska, 2022a, 2022b) and in the processing of LiDAR data (Duchnowski and Wyszkowska, 2022c).

This approach to processing observations is also suitable for geodesy if there are at least two subsets of observations, one from the first epoch and the other from the second epoch. There is no need to specify the order of the observations in advance – the observations can be interchanged arbitrarily, leading to the same result.

In this article we compare the squared and the absolute Msplit method and show the advantages and disadvantages of both methods. Using examples of 2D geodetic networks, we perform a comparative analysis of the results obtained with the two methods, which is the main contribution of the article.

2 SQUARED AND ABSOLUTE M_{SPLIT} METHOD

2.1 Comparing squared and absolutne M_{split} method

The derivation of the equations for the squared M_{split} method can be found in the literature (see the captions of Tables 1, 2 and 3) and is therefore not repeated in this article.

A comparison of the squared and absolute M_{split} methods is shown in Table 1. We will analyse two temporal epochs of observations, labelled 1 (first measurement) and 2 (second measurement).

Table 1.
A comparison of the squared and absolute M_{split} methods (Duchnowski and Wyszkowska, 2022a, 2022b, 2022c, 2024; Wyszkowska and Duchnowski, 2020, 2022a, 2022b; Wiśniewski, 2029a, 2009b, 2009c, 2010; Wiśniewski and Zienkiewicz, 2021)

Operation	**Squared M_{split} method (SMS)**	**Absolute M_{split} method (AMS)**
	Traditional iteration method	Parallel iteration method
Number of iterations	Usually less iterations needed	Usually more iterations needed
Sensitivity to gross errors	More sensitive	Less sensitive
Sensitivity to the different number of observations in subsets	Less sensitive	More sensitive
Decrease of accuracy with increasing observation standard deviations	Accuracy decreases faster	Accuracy decreases slower
Division of the functional model into two competitive functional models	$\mathbf{y}=\mathbf{A}\mathbf{x}+\mathbf{v}\overset{\text{split}}{\longrightarrow}\begin{cases}\mathbf{y}=\mathbf{A}\mathbf{x}_{(1)}+\mathbf{v}_{(1)}\\ \mathbf{y}=\mathbf{A}\mathbf{x}_{(2)}+\mathbf{v}_{(2)}\end{cases}$	
Objective function	$\varphi(\mathbf{x}_{(1)},\mathbf{x}_{(2)})=\sum_{i=1}^{n}v_{i(1)}^2v_{i(2)}^2$	$\varphi(\mathbf{x}_{(1)},\mathbf{x}_{(2)})=\sum_{i=1}^{n}\lvert v_{i(1)}\rvert\,\lvert v_{i(2)}\rvert$
Influence functions	$\begin{cases}\psi_{(1)}(v_{i(1)},v_{i(2)})=2v_{i(1)}v_{i(2)}^2\\ \psi_{(2)}(v_{i(1)},v_{i(2)})=2v_{i(1)}^2v_{i(2)}\end{cases}$	$\begin{cases}\psi_{(1)}(v_{i(1)},v_{i(2)})=\begin{cases}-\lvert v_{i(2)}\rvert \text{ for } v_{i(1)}<0\\ \lvert v_{i(2)}\rvert \text{ for } v_{i(1)}>0\end{cases}\\ \psi_{(2)}(v_{i(1)},v_{i(2)})=\begin{cases}-\lvert v_{i(1)}\rvert \text{ for } v_{i(2)}<0\\ \lvert v_{i(1)}\rvert \text{ for } v_{i(2)}>0\end{cases}\end{cases}$
Weight functions	$\begin{cases}w_{(1)}(v_{i(1)},v_{i(2)})=v_{i(2)}^2\\ w_{(2)}(v_{i(1)},v_{i(2)})=v_{i(1)}^2\end{cases}$	$\begin{cases}w_{(1)}(v_{i(1)},v_{i(2)})=\begin{cases}-\dfrac{\lvert v_{i(2)}\rvert}{2\lvert v_{i(1)}\rvert} \text{ for } v_{i(1)}<0\\ \dfrac{\lvert v_{i(2)}\rvert}{2\lvert v_{i(1)}\rvert} \text{ for } v_{i(1)}>0\end{cases}\\ w_{(2)}(v_{i(1)},v_{i(2)})=\begin{cases}-\dfrac{\lvert v_{i(1)}\rvert}{2\lvert v_{i(2)}\rvert} \text{ for } v_{i(2)}<0\\ \dfrac{\lvert v_{i(1)}\rvert}{2\lvert v_{i(2)}\rvert} \text{ for } v_{i(2)}>0\end{cases}\end{cases}$ To avoid singularity in denominator $\begin{cases}w_{(1)}^*(v_{i(1)},v_{i(2)})=\begin{cases}-\dfrac{\lvert v_{i(2)}\rvert}{2c^*} \text{ for } v_{i(1)}<c^*\\ \dfrac{\lvert v_{i(2)}\rvert}{2\lvert v_{i(1)}\rvert} \text{ for } v_{i(1)}\geq c^*\end{cases}\\ w_{(2)}^*(v_{i(1)},v_{i(2)})=\begin{cases}-\dfrac{\lvert v_{i(1)}\rvert}{2c^*} \text{ for } v_{i(2)}<c^*\\ \dfrac{\lvert v_{i(1)}\rvert}{2\lvert v_{i(2)}\rvert} \text{ for } v_{i(2)}\geq c^*\end{cases}\end{cases}$
Gradient	$\begin{cases}\mathbf{g}_{(1)}(\mathbf{x}_{(1)},\mathbf{x}_{(2)})=-2\mathbf{A}^T\mathbf{w}_{(1)}(\mathbf{v}_{(1)},\mathbf{v}_{(2)})\mathbf{v}_{(1)}\\ \mathbf{g}_{(2)}(\mathbf{x}_{(1)},\mathbf{x}_{(2)})=-2\mathbf{A}^T\mathbf{w}_{(2)}(\mathbf{v}_{(1)},\mathbf{v}_{(2)})\mathbf{v}_{(2)}\end{cases}$	
Hessian matrix	$\begin{cases}\mathbf{H}_{(1)}(\mathbf{x}_{(1)},\mathbf{x}_{(2)})=2\mathbf{A}^T\mathbf{w}_{(1)}(\mathbf{v}_{(1)},\mathbf{v}_{(2)})\mathbf{A}\\ \mathbf{H}_{(2)}(\mathbf{x}_{(1)},\mathbf{x}_{(2)})=2\mathbf{A}^T\mathbf{w}_{(2)}(\mathbf{v}_{(1)},\mathbf{v}_{(2)})\mathbf{A}\end{cases}$	
Weight	$\begin{cases}\mathbf{w}_{(1)}(\mathbf{v}_{(1)},\mathbf{v}_{(2)})=\mathrm{diag}[w_{(1)}(v_{1(1)},v_{1(2)}) \quad \cdots \quad w_{(1)}(v_{n(1)},v_{n(2)})]\\ \mathbf{w}_{(2)}(\mathbf{v}_{(1)},\mathbf{v}_{(2)})=\mathrm{diag}[w_{(2)}(v_{1(1)},v_{1(2)}) \quad \cdots \quad w_{(2)}(v_{n(1)},v_{n(2)})]\end{cases}$	

Explanation of symbols in Table 1:

y ... observation vector,
A ... coefficient matrix,
x ... unknown parameter vector,
v ... observation error vector,
$i = 1, \ldots, n$ ... observation counter,
n ... number of observations (both epochs),
$\varphi(\mathbf{x}_{(1)}, \mathbf{x}_{(2)}) = \sum_{i=1}^{n} \rho(v_{i(1)}, v_{i(2)}) = \sum_{i=1}^{n} \rho_{(1)}(v_{i(1)}, v_{i(2)}) \rho_{(2)}(v_{i(1)}, v_{i(2)})$ ... objective function,
$\rho(v_{i(1)}, v_{i(2)})$ ... objective function's component depending on a chosen variant,

$$\begin{cases} \psi_{(1)}(v_{i(1)}, v_{i(2)}) = \frac{\partial \rho_{(1)}(v_{i(1)}, v_{i(2)})}{\partial v_{i(1)}} \\ \psi_{(2)}(v_{i(1)}, v_{i(2)}) = \frac{\partial \rho_{(2)}(v_{i(1)}, v_{i(2)})}{\partial v_{i(2)}} \end{cases} \text{... influence functions,}$$

$$\begin{cases} w_{(1)}(v_{i(1)}, v_{i(2)}) \\ w_{(2)}(v_{i(1)}, v_{i(2)}) \end{cases} = \begin{matrix} \frac{\psi_{(1)}(v_{i(1)}, v_{i(2)})}{2 v_{i(1)}} \\ \frac{\psi_{(2)}(v_{i(1)}, v_{i(2)})}{2 v_{i(2)}} \end{matrix} \text{... weight functions,}$$

c^* ... small positive costant (e.g. 0.001),

$$\begin{cases} \mathbf{g}_{(1)}(\mathbf{x}_{(1)}, \mathbf{x}_{(2)}) = \left[\frac{\partial \varphi(\mathbf{x}_{(1)}, \mathbf{x}_{(2)})}{\partial \mathbf{x}_{(1)}} \right]^T \\ \mathbf{g}_{(2)}(\mathbf{x}_{(1)}, \mathbf{x}_{(2)}) = \left[\frac{\partial \varphi(\mathbf{x}_{(1)}, \mathbf{x}_{(2)})}{\partial \mathbf{x}_{(2)}} \right]^T \end{cases} \text{... gradient,}$$

$$\begin{cases} \mathbf{H}_{(1)}(\mathbf{x}_{(1)}, \mathbf{x}_{(2)}) = \frac{\partial^2 \varphi(\mathbf{x}_{(1)}, \mathbf{x}_{(2)})}{\partial \mathbf{x}_{(1)} \partial \mathbf{x}_{(1)}^T} \\ \mathbf{H}_{(2)}(\mathbf{x}_{(1)}, \mathbf{x}_{(2)}) = \frac{\partial^2 \varphi(\mathbf{x}_{(1)}, \mathbf{x}_{(2)})}{\partial \mathbf{x}_{(2)} \partial \mathbf{x}_{(2)}^T} \end{cases} \text{... Hessian matrix.}$$

2.2 Computation procedure

The solutions for both the squared and the absolute M_{split} method are calculated by an iterative process. The initial values of the corrections to the approximate values of the unknowns, in this case the corrections to the approximate coordinates, are calculated using the least squares adjustment (LSA) method. The iterative process is terminated when the differences between the corrections to the approximate coordinate values between two successive iterations are smaller than the selected threshold value ε.

In 1D geodetic networks, the mathematical model $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{v}$ represents a linear relationship between the observations **y** and the unknowns **x**. In 2D geodesic networks however, this relationship is non-linear. To deal with this, we linearise the model and express it in the familiar form $\mathbf{f} = \mathbf{A}\mathbf{x} + \mathbf{v}$, where **f** represents the residuals (Leick, 1980, p. 51).

In the system of correction equations $\mathbf{f} = \mathbf{A}\mathbf{x} + \mathbf{v}$, the elements of the matrix **A** for the observed direction are calculated, for example, using equations 7.51 (Mihailović, 1981, p. 313):

$v_{ri} = a_{ri} x_r + b_{ri} y_r + a_{ir} x_i + b_{ir} y_i + z_r + f_{ri}$, where

$a_{ri} = -\frac{\sin n_{ri}}{S_{ri}^0}$, $b_{ri} = \frac{\cos n_{ri}}{S_{ri}^0}$, $a_{ir} = -a_{ri}$, $b_{ir} = -b_{ri}$ are the elements of coeficient matrix **A**,

x_r, y_r, x_i, y_i and z_r are corrections to aproximate values of coordinates of points r and i and corrections to orientation angle on point r. They are all elements of corrections vector **x**, while the elements of the vector **f** are determined using equations 7.52 (Mihailović, 1981, p. 313):

$f_{ri} = n_{ri} + z_r^0 - \alpha_{ri}$,
n_{ri} is approximate direction angle between points r and i,
z_r^0 is approximate orientation angle on point r,
α_{ri} is measured direction from r towards i,
S_{ri}^0 is approximate distance between r and i.

In this system, the orientation unknowns must be eliminated. Gaussian elimination is used for this purpose. Similarly, in the system of correction equations $\mathbf{f} = \mathbf{Ax} + \mathbf{v}$, the elements of the matrix **A** for the measured distance are calculated using equations 8.37 (Mihailović, 1981, p. 408):

$v_{ri} = a_{ri}x_r + b_{ri}y_r + a_{ir}x_i + b_{ir}y_i + f_{ri}$, where

$a_{ri} = \cos n_{ri}$, $b_{ri} = \sin n_{ri}$, $a_{ir} = -a_{ri}$, $b_{ir} = -b_{ri}$ are the elements of coeficient matrix **A**,

and the elements of the vector **f** are determined using equations 8.37 (Mihailović, 1981, p. 408):

$f_{ri} = S_{ri}^0 - S_{ri}$, where
S_{ri} is measured distance between r and i.

Horizontal directions and slope distances are usually observed in 2D geodetic networks. These heterogeneous quantities usually have different degrees of accuracy, so we need to consider weights in the equations of both squared and absolute M_{split} methods (Zienkiewicz and Dabrowski, 2023; Zienkiewicz, 2019; Zienkiewicz, Hejbudzka and Dumalski, 2017; Zienkiewicz and Baryla, 2015) or use equivalent equations resulting from the application of Schreiber's third rule (Mihailović, 1981, p. 317).

For example, the matrix **A** and the vector **f** in the equations of the squared and absolute M_{split} method are constructed by first writing the elements in terms of the observed directions in the first epoch, followed by the elements in terms of the observed distances in the first epoch. The construction of the matrix **A** and vector **f** continues with elements relating to the observed directions and distances in the second epoch. Of course, we would obtain identical solutions for both the squared and absolute M_{split} methods, even if the order of the equations were completely rearranged.

Table 2.1.
Computation procedure of squared and absolute M_{split} methods (Duchnowski and Wyszkowska, 2022a, 2022b, 2022c; Wyszkowska and Duchnowski, 2019, 2020, 2022a; Wyszkowska, Duchnowski, Dumalski, 2021; Wiśniewski, 2009b, 2009c, 2010)

Operation	Squared M_{split} method (SMS)
Correction of approximate values of unknowns (Coordinates)	$\begin{cases} \mathbf{x}_{(1)}^{j} = -\left[\mathbf{H}_{(1)}\left(\mathbf{x}_{(1)}^{j-1}, \mathbf{x}_{(2)}^{j-1}\right)\right]^{-1} \mathbf{g}_{(1)}\left(\mathbf{x}_{(1)}^{j-1}, \mathbf{x}_{(2)}^{j-1}\right) \\ \mathbf{x}_{(2)}^{j} = -\left[\mathbf{H}_{(2)}\left(\mathbf{x}_{(1)}^{j}, \mathbf{x}_{(2)}^{j-1}\right)\right]^{-1} \mathbf{g}_{(2)}\left(\mathbf{x}_{(1)}^{j}, \mathbf{x}_{(2)}^{j-1}\right) \end{cases}$
New values of observation errors	$\begin{cases} \mathbf{v}_{(1)}^{j} = \mathbf{f} - \mathbf{A}\mathbf{x}_{(1)}^{j} \\ \mathbf{v}_{(2)}^{j} = \mathbf{f} - \mathbf{A}\mathbf{x}_{(2)}^{j} \end{cases}$
Gradient	$\begin{cases} \mathbf{g}_{(1)}\left(\mathbf{x}_{(1)}^{j-1}, \mathbf{x}_{(2)}^{j-1}\right) = -2\mathbf{A}^{T}\mathbf{w}_{(1)}\left(\mathbf{v}_{(1)}^{j-1}, \mathbf{v}_{(2)}^{j-1}\right)\mathbf{v}_{(1)}^{j-1} \\ \mathbf{g}_{(2)}\left(\mathbf{x}_{(1)}^{j}, \mathbf{x}_{(2)}^{j-1}\right) = -2\mathbf{A}^{T}\mathbf{w}_{(2)}\left(\mathbf{v}_{(1)}^{j}, \mathbf{v}_{(2)}^{j-1}\right)\mathbf{v}_{(2)}^{j-1} \end{cases}$
Hessian matrix	$\begin{cases} \mathbf{H}_{(1)}\left(\mathbf{x}_{(1)}^{j-1}, \mathbf{x}_{(2)}^{j-1}\right) = 2\mathbf{A}^{T}\mathbf{w}_{(1)}\left(\mathbf{v}_{(1)}^{j-1}, \mathbf{v}_{(2)}^{j-1}\right)\mathbf{A} \\ \mathbf{H}_{(2)}\left(\mathbf{x}_{(1)}^{j}, \mathbf{x}_{(2)}^{j-1}\right) = 2\mathbf{A}^{T}\mathbf{w}_{(2)}\left(\mathbf{v}_{(1)}^{j}, \mathbf{v}_{(2)}^{j-1}\right)\mathbf{A} \end{cases}$
Weight	$\begin{cases} \mathbf{w}_{(1)}\left(\mathbf{v}_{(1)}^{j-1}, \mathbf{v}_{(2)}^{j-1}\right) = \mathbf{w}_{(1)}\left(\mathbf{v}_{(2)}^{j-1}\right) \\ \mathbf{w}_{(2)}\left(\mathbf{v}_{(1)}^{j}, \mathbf{v}_{(2)}^{j-1}\right) = \mathbf{w}_{(2)}\left(\mathbf{v}_{(1)}^{j}\right) \end{cases}$ $\begin{cases} \mathbf{w}_{(1)}\left(\mathbf{v}_{(2)}^{j-1}\right) = \mathrm{diag}\left[\left(v_{1(2)}^{j-1}\right)^{2} \quad \cdots \quad \left(v_{n(2)}^{j-1}\right)^{2}\right] \\ \mathbf{w}_{(2)}\left(\mathbf{v}_{(1)}^{j}\right) = \mathrm{diag}\left[\left(v_{1(1)}^{j}\right)^{2} \quad \cdots \quad \left(v_{n(1)}^{j}\right)^{2}\right] \end{cases}$

Table 2.2.
Computation procedure of squared and absolute M_{split} methods (Duchnowski and Wyszkowska, 2022a, 2022b, 2022c; Wyszkowska and Duchnowski, 2019, 2020, 2022a; Wyszkowska, Duchnowski, Dumalski, 2021; Wiśniewski, 2009b, 2009c, 2010)

Operation	Absolute M_{split} method (AMS)
Correction of approximate values of unknowns (Coordinates)	$\begin{cases} \mathbf{x}_{(1)}^{j} = -\left[\mathbf{H}_{(1)}\left(\mathbf{x}_{(1)}^{j-1}, \mathbf{x}_{(2)}^{j-1}\right)\right]^{-1} \mathbf{g}_{(1)}\left(\mathbf{x}_{(1)}^{j-1}, \mathbf{x}_{(2)}^{j-1}\right) \\ \mathbf{x}_{(2)}^{j} = -\left[\mathbf{H}_{(2)}\left(\mathbf{x}_{(1)}^{j-1}, \mathbf{x}_{(2)}^{j-1}\right)\right]^{-1} \mathbf{g}_{(2)}\left(\mathbf{x}_{(1)}^{j-1}, \mathbf{x}_{(2)}^{j-1}\right) \end{cases}$
New values of observation errors	$\begin{cases} \mathbf{v}_{(1)}^{j} = \mathbf{f} - \mathbf{A}\mathbf{x}_{(1)}^{j} \\ \mathbf{v}_{(2)}^{j} = \mathbf{f} - \mathbf{A}\mathbf{x}_{(2)}^{j} \end{cases}$
Gradient	$\begin{cases} \mathbf{g}_{(1)}\left(\mathbf{x}_{(1)}^{j-1}, \mathbf{x}_{(2)}^{j-1}\right) = -2\mathbf{A}^{T}\mathbf{w}_{(1)}\left(\mathbf{v}_{(1)}^{j-1}, \mathbf{v}_{(2)}^{j-1}\right)\mathbf{v}_{(1)}^{j-1} \\ \mathbf{g}_{(2)}\left(\mathbf{x}_{(1)}^{j-1}, \mathbf{x}_{(2)}^{j-1}\right) = -2\mathbf{A}^{T}\mathbf{w}_{(2)}\left(\mathbf{v}_{(1)}^{j-1}, \mathbf{v}_{(2)}^{j-1}\right)\mathbf{v}_{(2)}^{j-1} \end{cases}$
Hessian matrix	$\begin{cases} \mathbf{H}_{(1)}\left(\mathbf{x}_{(1)}^{j-1}, \mathbf{x}_{(2)}^{j-1}\right) = 2\mathbf{A}^{T}\mathbf{w}_{(1)}\left(\mathbf{v}_{(1)}^{j-1}, \mathbf{v}_{(2)}^{j-1}\right)\mathbf{A} \\ \mathbf{H}_{(2)}\left(\mathbf{x}_{(1)}^{j-1}, \mathbf{x}_{(2)}^{j-1}\right) = 2\mathbf{A}^{T}\mathbf{w}_{(2)}\left(\mathbf{v}_{(1)}^{j-1}, \mathbf{v}_{(2)}^{j-1}\right)\mathbf{A} \end{cases}$
Weight	$\mathbf{w}_{(1)}\left(\mathbf{v}_{(1)}^{j-1}, \mathbf{v}_{(2)}^{j-1}\right) =$ $= \mathrm{diag}\left[\begin{cases} -\dfrac{\left\lvert v_{1(2)}^{j-1}\right\rvert}{2c^{*}} \text{ for } v_{1(1)}^{j-1} < c^{*} \\ \dfrac{\left\lvert v_{1(2)}^{j-1}\right\rvert}{2\left\lvert v_{1(1)}^{j-1}\right\rvert} \text{ for } v_{1(1)}^{j-1} \geq c^{*} \end{cases} \cdots \begin{cases} -\dfrac{\left\lvert v_{n(2)}^{j-1}\right\rvert}{2c^{*}} \text{ for } v_{n(1)}^{j-1} < c^{*} \\ \dfrac{\left\lvert v_{n(2)}^{j-1}\right\rvert}{2\left\lvert v_{n(1)}^{j-1}\right\rvert} \text{ for } v_{n(1)}^{j-1} \geq c^{*} \end{cases}\right]$ $\mathbf{w}_{(2)}\left(\mathbf{v}_{(1)}^{j-1}, \mathbf{v}_{(2)}^{j-1}\right) =$ $= \mathrm{diag}\left[\begin{cases} -\dfrac{\left\lvert v_{1(1)}^{j-1}\right\rvert}{2c^{*}} \text{ for } v_{1(2)}^{j-1} < c^{*} \\ \dfrac{\left\lvert v_{1(1)}^{j-1}\right\rvert}{2\left\lvert v_{1(2)}^{j-1}\right\rvert} \text{ for } v_{1(2)}^{j-1} \geq c^{*} \end{cases} \cdots \begin{cases} -\dfrac{\left\lvert v_{n(1)}^{j-1}\right\rvert}{2c^{*}} \text{ for } v_{n(2)}^{j-1} < c^{*} \\ \dfrac{\left\lvert v_{n(1)}^{j-1}\right\rvert}{2\left\lvert v_{n(2)}^{j-1}\right\rvert} \text{ for } v_{n(2)}^{j-1} \geq c^{*} \end{cases}\right]$

Explanation of symbols in Tables 2.1 and 2.2:

$j = 1, \ldots, k$ ... iteration counter,

k ... the number of iterations required to compute the solutions of the squared and absolute M_{split} method,

c^* ... small positive constant (e. g. 0.001), to avoid singularity in the denominator.

The calculation of the solution, i.e., the calculation of the vector $\mathbf{x}$, begins by computing the corrections to the approximate coordinate values $\mathbf{x}_{(\mathrm{LSA})}$ and the corrections to the observations $\mathbf{v}_{(\mathrm{LSA})}$ using the least squares adjustment (LSA) method. Then, in the iterative process, new values of the unknowns $\mathbf{x}_{(1)}^{j}$ in $\mathbf{v}_{(1)}^{j}$ as well as $\mathbf{x}_{(2)}^{j}$ in $\mathbf{v}_{(2)}^{j}$, are calculated. The iterative process is terminated when the changes in the unknown values $\mathbf{x}_{(1)}$ between two consecutive iterations are smaller than a selected threshold. The same applies to the values of $\mathbf{x}_{(2)}$. The calculation procedure is shown in Table 3.1 and 3.2.

Table 3.1.
Computation flow of squared and absolute M_{split} methods (Wiśniewski, 2009c).

Operation	Squared M_{split} method (SMS)
Least square method adjustment	$\mathbf{x}_{(\mathrm{LSA})} = (\mathbf{A}^{\mathbf{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathbf{T}}\mathbf{f}$ $\mathbf{v}_{(\mathrm{LSA})} = \mathbf{f} - \mathbf{A}\mathbf{x}_{(\mathrm{LSA})}$
Initial values of first subset in iteration process	$\mathbf{x}_{(1)}^{0} = \mathbf{x}_{(\mathrm{LSA})}$ $\mathbf{v}_{(1)}^{0} = \mathbf{v}_{(\mathrm{LSA})}$
Weight	$\mathbf{w}_{(2)}(\mathbf{v}_{(\mathrm{LSA})}) = \mathrm{diag}[v_{1(\mathrm{LSA})}^{2} \quad \cdots \quad v_{n(\mathrm{LSA})}^{2}]$
Initial values of second subset in iteration process	$\mathbf{x}_{(2)}^{0} = \mathbf{x}_{(\mathrm{LSA})} + \left[\mathbf{A}^{T}\mathbf{w}_{(2)}(\mathbf{v}_{(\mathrm{LSA})})\mathbf{A}\right]^{-1}\mathbf{A}^{T}\mathbf{w}_{(2)}(\mathbf{v}_{(\mathrm{LSA})})\mathbf{f}$ $\mathbf{v}_{(2)}^{0} = \mathbf{f} - \mathbf{A}\mathbf{x}_{(2)}^{0}$
Next steps in iteration process	$\mathbf{w}_{(1)}(\mathbf{v}_{(2)}^{j-1}) = \mathrm{diag}\left[(v_{1(2)}^{j-1})^{2} \quad \cdots \quad (v_{n(2)}^{j-1})^{2}\right]$ $\mathbf{x}_{(1)}^{j} = \mathbf{x}_{(1)}^{j-1} + \left[\mathbf{A}^{T}\mathbf{w}_{(1)}(\mathbf{v}_{(2)}^{j-1})\mathbf{A}\right]^{-1}\mathbf{A}^{T}\mathbf{w}_{(1)}(\mathbf{v}_{(2)}^{j-1})\mathbf{v}_{(1)}^{j-1}$ $\mathbf{v}_{(1)}^{j} = \mathbf{f} - \mathbf{A}\mathbf{x}_{(1)}^{j}$ $\mathbf{w}_{(2)}(\mathbf{v}_{(1)}^{j}) = \mathrm{diag}\left[(v_{1(1)}^{j})^{2} \quad \cdots \quad (v_{n(1)}^{j})^{2}\right]$ $\mathbf{x}_{(2)}^{j} = \mathbf{x}_{(1)}^{j} + \left[\mathbf{A}^{T}\mathbf{w}_{(2)}(\mathbf{v}_{(1)}^{j})\mathbf{A}\right]^{-1}\mathbf{A}^{T}\mathbf{w}_{(2)}(\mathbf{v}_{(1)}^{j})\mathbf{v}_{(2)}^{j-1}$ $\mathbf{v}_{(2)}^{j} = \mathbf{f} - \mathbf{A}\mathbf{x}_{(2)}^{j}$
Termination condition of the iteration process	$\left\|\mathbf{x}_{(1)}^{j-1} - \mathbf{x}_{(1)}^{j}\right\| < \varepsilon$ and $\left\|\mathbf{x}_{(2)}^{j-1} - \mathbf{x}_{(2)}^{j}\right\| < \varepsilon$

Table 3.2.
Computation flow of squared and absolute M_{split} methods (Wiśniewski, 2009c).

Operation	Absolute M_{split} method (AMS)
Least square method adjustment	$\mathbf{x}_{(\mathrm{LSA})} = (\mathbf{A}^{\mathbf{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathbf{T}}\mathbf{f}$ $\mathbf{v}_{(\mathrm{LSA})} = \mathbf{f} - \mathbf{A}\mathbf{x}_{(\mathrm{LSA})}$
Initial values of first subset in iteration process	$\mathbf{x}_{(1)}^{0} = \mathbf{x}_{(\mathrm{LSA})}$ $\mathbf{v}_{(1)}^{0} = \mathbf{v}_{(\mathrm{LSA})}$
Weight	$\mathbf{w}_{(2)}\left(\mathbf{v}_{(\mathrm{LSA})}, \mathbf{v}_{(1)}^{j-1}\right) =$ $= \mathrm{diag}\left[\begin{cases} -\dfrac{\lvert v_{1(\mathrm{LSA})}\rvert}{2c^*} \text{ for } v_{1(1)} < c^* \\ \dfrac{\lvert v_{1(\mathrm{LSA})}\rvert}{2\lvert v_{1(1)}\rvert} \text{ for } v_{1(1)} \geq c^* \end{cases} \cdots \begin{cases} -\dfrac{\lvert v_{n(\mathrm{LSA})}\rvert}{2c^*} \text{ for } v_{n(1)} < c^* \\ \dfrac{\lvert v_{n(\mathrm{LSA})}\rvert}{2\lvert v_{n(1)}\rvert} \text{ for } v_{n(1)} \geq c^* \end{cases}\right]$
Initial values of second subset in iteration process	$\mathbf{x}_{(2)}^{0} = \mathbf{x}_{(\mathrm{LSA})} + \left[\mathbf{A}^T\mathbf{w}_{(2)}\left(\mathbf{v}_{(\mathrm{LSA})}\right)\mathbf{A}\right]^{-1}\mathbf{A}^T\mathbf{w}_{(2)}\left(\mathbf{v}_{(\mathrm{LSA})}\right)\mathbf{f}$ $\mathbf{v}_{(2)}^{0} = \mathbf{f} - \mathbf{A}\mathbf{x}_{(2)}^{0}$
Next steps in iteration process	$\mathbf{w}_{(1)}\left(\mathbf{v}_{(1)}^{j-1}, \mathbf{v}_{(2)}^{j-1}\right) =$ $= \mathrm{diag}\left[\begin{cases} -\dfrac{\lvert v_{1(2)}^{j-1}\rvert}{2c^*} \text{ for } v_{1(1)}^{j-1} < c^* \\ \dfrac{\lvert v_{1(2)}^{j-1}\rvert}{2\lvert v_{1(1)}^{j-1}\rvert} \text{ for } v_{1(1)}^{j-1} \geq c^* \end{cases} \cdots \begin{cases} -\dfrac{\lvert v_{n(2)}^{j-1}\rvert}{2c^*} \text{ for } v_{n(1)}^{j-1} < c^* \\ \dfrac{\lvert v_{n(2)}^{j-1}\rvert}{2\lvert v_{n(1)}^{j-1}\rvert} \text{ for } v_{n(1)}^{j-1} \geq c^* \end{cases}\right]$ $\mathbf{x}_{(1)}^{j} = \mathbf{x}_{(1)}^{j-1} + \left[\mathbf{A}^T\mathbf{w}_{(1)}\left(\mathbf{v}_{(1)}^{j-1}, \mathbf{v}_{(2)}^{j-1}\right)\mathbf{A}\right]^{-1}\mathbf{A}^T\mathbf{w}_{(1)}\left(\mathbf{v}_{(1)}^{j-1}, \mathbf{v}_{(2)}^{j-1}\right)\mathbf{v}_{(1)}^{j-1}$ $\mathbf{v}_{(1)}^{j} = \mathbf{f} - \mathbf{A}\mathbf{x}_{(1)}^{j}$ $\mathbf{w}_{(2)}\left(\mathbf{v}_{(1)}^{j-1}, \mathbf{v}_{(2)}^{j-1}\right) =$ $= \mathrm{diag}\left[\begin{cases} -\dfrac{\lvert v_{1(1)}^{j-1}\rvert}{2c^*} \text{ for } v_{1(2)}^{j-1} < c^* \\ \dfrac{\lvert v_{1(1)}^{j-1}\rvert}{2\lvert v_{1(2)}^{j-1}\rvert} \text{ for } v_{1(2)}^{j-1} \geq c^* \end{cases} \cdots \begin{cases} -\dfrac{\lvert v_{n(1)}^{j-1}\rvert}{2c^*} \text{ for } v_{n(2)}^{j-1} < c^* \\ \dfrac{\lvert v_{n(1)}^{j-1}\rvert}{2\lvert v_{n(2)}^{j-1}\rvert} \text{ for } v_{n(2)}^{j-1} \geq c^* \end{cases}\right]$ $\mathbf{x}_{(2)}^{j} = \mathbf{x}_{(1)}^{j-1} + \left[\mathbf{A}^T\mathbf{w}_{(2)}\left(\mathbf{v}_{(1)}^{j-1}, \mathbf{v}_{(2)}^{j-1}\right)\mathbf{A}\right]^{-1}\mathbf{A}^T\mathbf{w}_{(2)}\left(\mathbf{v}_{(1)}^{j-1}, \mathbf{v}_{(2)}^{j-1}\right)\mathbf{v}_{(2)}^{j-1}$ $\mathbf{v}_{(2)}^{j} = \mathbf{f} - \mathbf{A}\mathbf{x}_{(2)}^{j}$
Termination condition of the iteration process	$\left\lVert \mathbf{x}_{(1)}^{j-1} - \mathbf{x}_{(1)}^{j} \right\rVert < \varepsilon$ and $\left\lVert \mathbf{x}_{(2)}^{j-1} - \mathbf{x}_{(2)}^{j} \right\rVert < \varepsilon$

Explanation of symbols in Tables 3.1 and 3.2:

ε ... criterion of convergence for termination of iteration process (e.g. 0.001),
c^* ... small positive constant (e. g. 0.001), to avoid singularity in the denominator.

At the end of the iterative process, we obtain the results $\mathbf{x}_{(1)}^{k}$ and $\mathbf{x}_{(2)}^{k}$ for the squared or absolute M_{split} method. The difference of the coordinates from the last iteration k between the two subsets for each point represents the displacement of the point:

$\mathbf{p} = \mathbf{x}_{(2)}^{k} - \mathbf{x}_{(1)}^{k}$.

3 CALCULATION EXAMPLE

The results of the squared or absolute M_{split} method are compared in two case studies from the literature, which is the main purpose of this article. In the first case, we consider a combined geodetic network where both directions and distances have been simulated. In the second case, we analyse a triangulation network where the directions were observed. In both cases, two measurement epochs are used, with the same number of directions and distances simulated/observed in each epoch. There are no outliers in either geodetic network.

3.1 Geodetic hexagon

The geodetic network is published in Mihailović and Aleksić, 1994, p. 173, and the network diagram is shown in Figure 2. As is customary in this paper, we will present the results in Table 4.

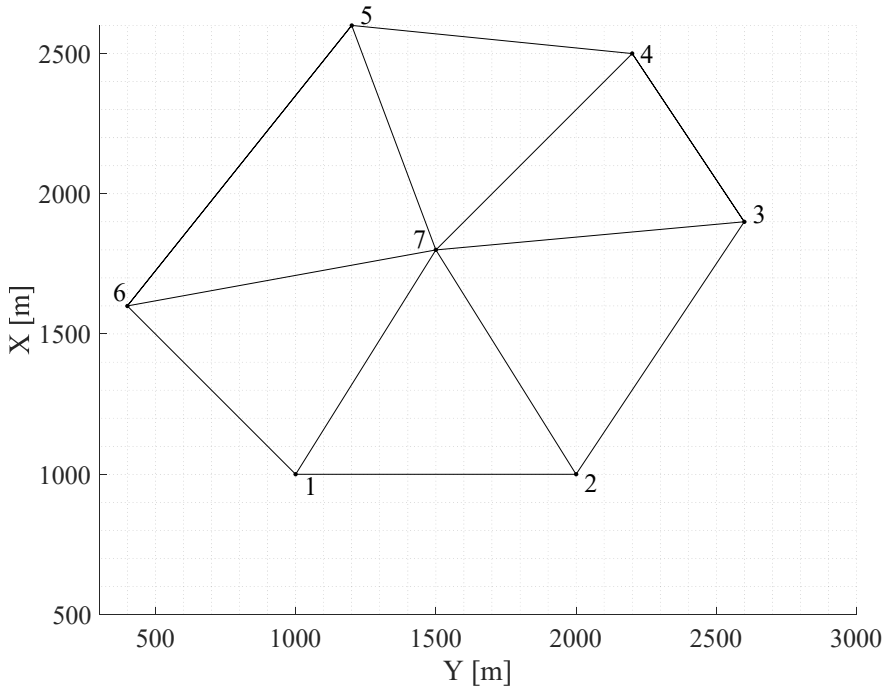

Figure 2. Geodetic hexagon.

Table 4.
Comparison of the results of squared and absolute M_{split} method – hexagon

Parameter	Squared M_{split} method (SMS)			Absolute M_{split} method (AMS)		
Number of points	7					
Number of horizontal directions	24					
Number of distances	24					
Displacement simulation	Point	*pY* [mm]	*pX* [mm]			
	1	-20.0	-34.6			
	2	-30.0	52.0			
	3	25.0	-43.3			
	4	0.0	0.0			
	5	0.0	0.0			
	6	0.0	0.0			
	7	25.0	43.3			
Aproximate coordinates	Point	*Y* [m]	*X* [m]			
	1	1000	1000			
	2	2000	1000			
	3	2600	1900			
	4	2200	2500			
	5	1200	2600			
	6	400	1600			
	7	1500	1800			
Number of iteration steps	9			36		
Point displacement	Point	*pY* [mm]	*pX* [mm]	Point	*pY* [mm]	*pX* [mm]
	1	-8.5	-43.7	1	-10.6	-38.0
	2	-30.7	49.9	2	-32.4	56.7
	3	19.0	-33.5	3	22.2	-37.0
	4	-8.4	6.0	4	3.4	4.6
	5	0.7	-9.0	5	-10.0	-14.0
	6	1.7	-9.8	6	-0.2	-15.4
	7	26.3	40.1	7	27.6	42.9

In Figure 3, the calculated displacement using the squared and absolute M_{split} methods, along with the simulated displacement for comparison is presented.

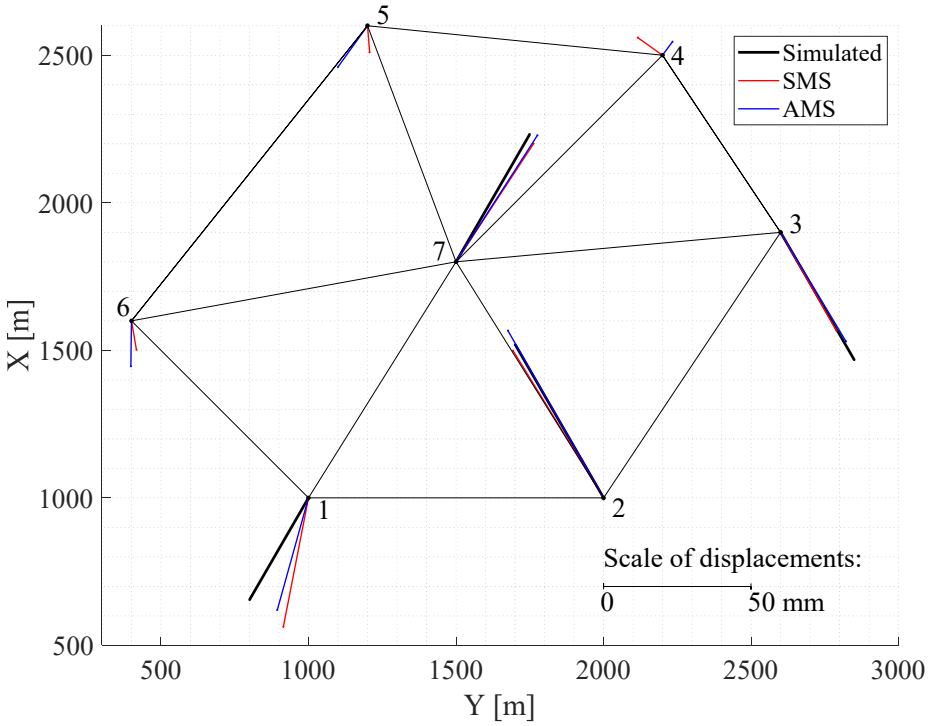

Figure 3. The calculated displacement using the squared and absolute M_{split} methods, along with the simulated displacement (red - squared M_{split} method, blue - absolute M_{split} method, black - simulated displacement).

According to the simulated displacements, the results of the squared M_{split} method differ the most by 11.5 mm (Y coordinate of point 1) and 9.8 mm (X coordinates of points 3 and 6). The results of the absolute M_{split} method differ the most by 10.0 mm (Y coordinate of point 5) and 15.4 mm (X coordinate of point 6). When comparing the results of the squared and absolute M_{split} methods, the largest difference is 11.8 mm (Y coordinate of point 4) and 6.8 mm (X coordinate of point 2).

From Figure 3 we can see that the results of the squared and absolute methods are very similar for points with current displacement and differ more for stable points. The same is true for the difference between the simulated values and the results of both methods. The actual values of the average difference between the solutions can be found in Table 5.

Table 5. Average displacement difference for the first case

Difference	Stable points [mm]	Displaced points [mm]	All points [mm]
SMS – simulated	5.9	5.5	5.7
AMS – simulated	7.9	4.0	5.7
AMS – SMS	6.1	3.4	4.5

The number of iterations required to compute the absolute M_{split} solutions is four times higher than for squared M_{split}.

3.2 Geodetic network „Tušanj“

Geodetic network is published in (Vrce, 2011), p. 20-21, and presented in Figure 4. Computation results are in Table 6.

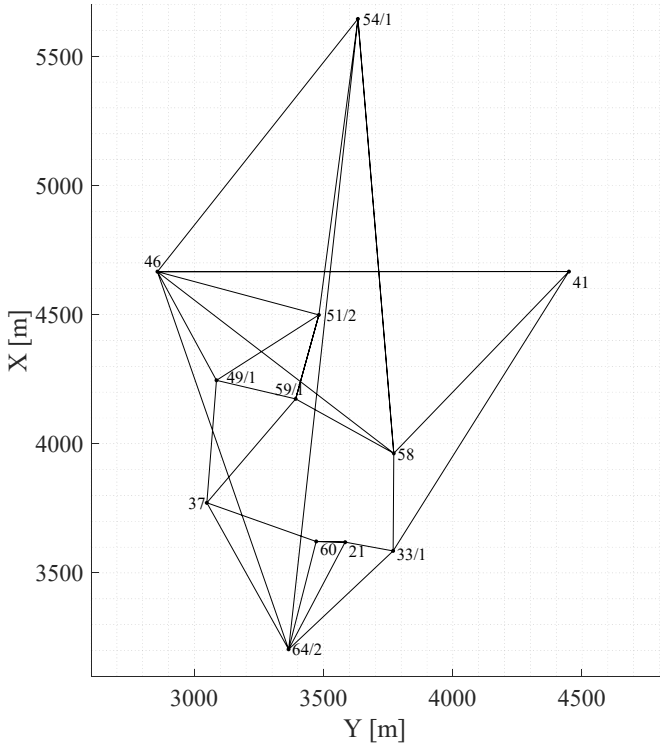

Figure 4. Geodetic network „Tušanj“.

Table 6.
Comparison of the results of squared and absolute M_{split} method - „Tušanj“.

Parameter/Operation	**Squared M_{split} method (SMS)**			**Absolute M_{split} method (AMS)**		
Number of points	12					
Number of horizontal directions	50 (0. epoch), 53 (4. epoch)					
Aproximate coordinates	Point	*Y* [m]	*X* [m]			
	37	3048.074	3771.266			
	51/2	3481.550	4498.590			
	54/1	3632.625	5644.253			
	41	4449.397	4666.272			
	46	2856.838	4666.189			
	64/2	3364.375	3204.254			
	21	3583.462	3618.911			
	58	3771.560	3962.764			
	60	3471.440	3621.637			
	33/1	3768.861	3585.168			
	49/1	3085.371	4245.558			
	59/1	3391.880	4173.541			
Number of iterations	10			26		
Point displacements	Point	*pY* [mm]	*pX* [mm]	Point	*pY* [mm]	*pX* [mm]
	37	-2.5	38.3	37	-1.1	37.2
	51/2	-28.7	-17.1	51/2	-29.9	-25.1
	54/1	-2.9	4.9	54/1	1.2	3.5
	41	36.7	-4.7	41	48.6	12.6
	46	-35.0	0.3	46	-32.5	-8.4
	64/2	-5.1	5.1	64/2	-9.7	8.8
	21	7.9	30.5	21	2.5	32.1
	58	4.0	-62.3	58	-0.2	-61.4
	60	11.5	36.7	60	7.2	38.0
	33/1	4.8	14.0	33/1	1.1	15.7
	49/1	19.2	-28.8	49/1	21.3	-33.3
	59/1	-10.0	-16.9	59/1	-8.5	-19.8

Figure 5 represents the displacements determined by squared and absolute M_{split} method.

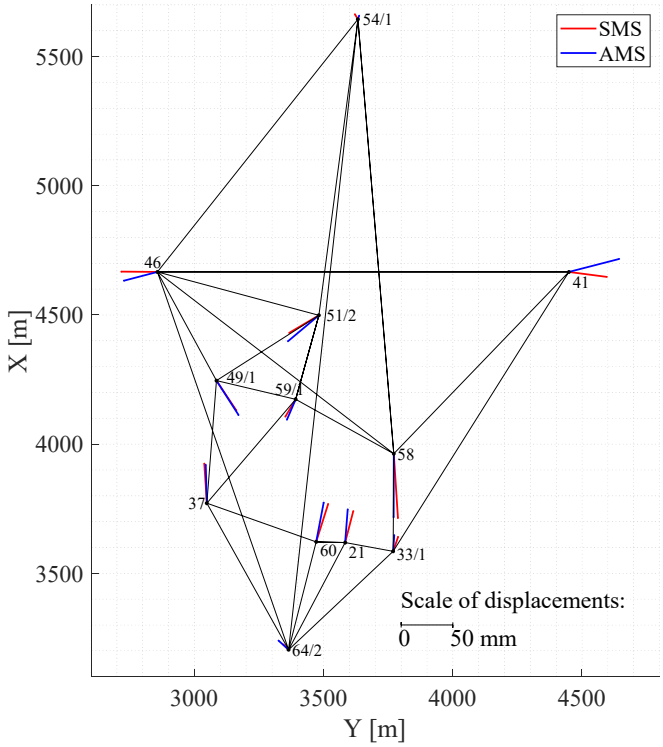

Figure 5. Displacements determined by squared and absolute M_{split} method (red - squared M_{split} method, blue - absolute M_{split} method).

The results calculated using the squared and absolute M_{split} methods cannot be compared with the simulated displacements, as these are not available - the network in question is a triangulation network where the directions were actually observed. However, when comparing the results of the squared and absolute M_{split} methods, the largest differences are 11.9 mm (*Y* coordinate of point 41) and 17.3 mm (*X* coordinate of point 41). The average difference between the squared and absolute M_{split} coordinates was 4.2 mm.

Number of iterations required to compute the absolute M_{split} solutions was 2.6 times higher as for the squared M_{split} method.

4 CONCLUSION

This paper presents the squared and absolute M_{split} methods, which are two variants of deformation analysis methods. These methods can be used to determine point displacements in geodetic networks.

The absolute M_{split} method is an alternative for a robust M-estimation. With the squared M_{split} method, gross errors in the observations are likely to lead to inaccurate results, whereas with the

absolute M_{split} method such errors have minimal impact. The disadvantage of the absolute method is generally the higher number of iterations required, but for the tasks considered in this article (small number of points) this is not a problem at all.

In this article, the squared and absolute M_{split} methods are presented with their respective equations and their similarities and differences are clearly illustrated in tables.

Subsequently, both methods were applied to two cases of 2D geodesic networks: one where directions and distances were simulated and another where directions were actually observed.

In order to compare the two M_{split} methods, which is the purpose of this article, a computer programme was developed and used to calculate all the results presented.

As expected, the comparison of the recorded results confirmed that both methods gave quite similar results. In the first case with known defects, the average difference between the known coordinates and the results of both M_{split} methods was 5.7 mm. For the second case, there were no known displacements, but the average distance between the squared and absolute solutions was 4.5 mm.

For the first geodesic network, where the observations were simulated, we can also say that the results are comparable to those obtained with other methods for deformation analysis as described in Ambrožič et al., 2019, and Hamza et al., 2020. For the second geodetic network, the results are comparable to the results of the Hannover deformation analysis described in (Vrce, 2011).

A limitation of both the squared and absolute M_{split} methods is that they do not allow the determination of statistically significant displacements as they do not calculate test statistics.

To summarise, both the squared and absolute M_{split} methods are useful for determining point displacements, even in 2D geodesic networks (their applicability for these purposes in 1D elevation networks is demonstrated in the cited literature).

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